AXIALLY SYMMETRICAL INSTABILITY MODES IN A CYLINDRICAL SHELL UNDER IMPACT

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An analysis is presented of the interaction between longitudinal and transverse motions of a circular cylindrical shell under impact on the end surface. At infinite and finite velocities of perturbation propagation along the generatrix this analysis reveals the instability modes in the shell which build up fastest and are similar to those revealed if the buckling process at a finite velocity of perturbation propagation were described in the real time of compressive loading action. It is established that a cylindrical shell under intensive loading can be simulated by a rod under longitudinal impact (the similarity parameters are indicated). This conclusion is confirmed by a comparison with experimental results.

Elastic systems are characterized by a selective amplification of certain higher-order instability modes under conditions of high-intensity loading [1]. The effect of a wave process on the buckling of rods and shells with a sudden application of a load to an elastic system has been observed in the experiments in [2-4]. The instability mode has been determined in an asymptotic representation for a semiinfinite rod, assuming a finite velocity of longitudinal perturbations [5] and with the aid of series expansion on a variable interval [6]. A problem analogous to the one which will be considered here has been solved numerically in [7]; probably because of the low impact rate, no wave generation was observed along the shell. An effect of a wave process on the buckling mode is mentioned in [8].

1. Formulation of the Problem. The longitudinal and the transverse motions of a circular cylindrical shell are described by the following system of equations:

$$D\left(w_{xxxx}+12\frac{w}{R^2h^2}\right)-\frac{\nu Eh}{R(1-\nu^2)}u_x+(Nw_x)_x+\rho h w_{tt}=f(x,t) \tag{1.1}$$

$$N_{xx}-c^{-2}N_{tt}=\nu\rho(h/R)w_{tt} \tag{1.2}$$

Here x, t are the longitudinal coordinate and the time; u, w are the longitudinal and the transverse displacement of the mean shell surface with the radius R; the subscripts refer to differentiation with respect to the respective variables; E, ν, ρ, are the Young modulus, the Poisson ratio, and the material density; D is the cylindrical rigidity; $c=\{E/[\rho(1-\nu^2)]\}^{1/2}$ is the velocity of sound; $f(x,t)$ is a function defined by perturbations or imperfections; and N is the longitudinal force, defined in the linear approximation by the equation

$$N=[Eh/(1-\nu^2)]\,(u_x-\nu w/R) \tag{1.3}$$

The initial and the boundary conditions for the impact state at the time of loading a semiinfinite hinge-supported shell which had been at rest before the impact are

$$\begin{gathered} w=w_t=0 \quad (t=0,0\leqslant x<\infty),\ w=w_{xx}=0 \quad (x=0) \\ N=N_0=\text{const} \quad (t>0,x=0),\quad N=N_t=0 \quad (t=0,x>0) \end{gathered} \tag{1.4}$$

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During the loading of the shell there may appear small transverse loads, the force N_0 may be applied eccentrically, and also the shape of the mean shell surface may differ from an ideal cylindrical one. All this is accounted for by function $f(x, t)$, which will be assumed given.

Equations (1.1) and (1.2) are identical to those known in § 215 [9] from the theory of vibrations of circular cylindrical shells, where the expressions for the forces at the mean surface have been linearized through superpositions.

The solution of problem (1.1), (1.2), (1.4) becomes much simpler when the longitudinal force $N(x, t) =$ const. This can be realized in two cases.

1) If the shell wall is sufficiently thin ($h/R \ll 1$) to make the expression on the right-hand side of (1.2) negligibly small; then the solution to the simplified wave equation with respect to the longitudinal force for the shell under conditions (1.4) becomes

$$N = N_0 = \text{const}$$

2) If the perturbations are propagated along the shell at an infinite velocity; then we have from the equation of motion

$$u_{xx} - \nu w_x / R = u_{tt} / c^2$$

that $N = N_0 =$ const at $c \to \infty$ (see (1.3)).

2. Analysis of Buckling Modes under a Constant Force. We consider the equation of dynamic instability for a shell:

$$D\,[w_{xxxx} + 12(1-\nu^2)R^{-2}h^{-2}w] + N_0 w_{xx} + \rho h w_{tt} = f^*(x, t) \tag{2.1}$$

This equation has been obtained by eliminating u_x from (1.1) with the aid of (1.3), and $f^*(x, t)$ denotes those components which are independent of w.

The initial and the boundary conditions for Eq. (2.1) are (L is the shell length)

$$w = w_t = 0 \quad (t = 0, 0 \leqslant x \leqslant L), \quad w = w_{xx} = 0 \quad (x = 0, L) \tag{2.2}$$

The solution to problem (2.1), (2.2) is sought in the form

$$w = \sum_{m=1}^{\infty} q_m(t)\, W_m(x), \quad W_m(x) = \sin\frac{m\pi x}{L} \quad (m = 1, 2, \ldots) \tag{2.3}$$

Here W_m are the instability modes.

We now analyze the behavior of a shell under intensive loading, i.e., under $N_0 > N^*$ with $N^* = 4[3(1-\nu^2)]^{1/2} DR^{-1}h^{-1}$ being the critical Euler load.

Of interest during the action of high-intensity loads $N_0 > N^*$ are those degrees of freedom in the system (instability modes) which correspond to a fast increase of deflections [1]. The fastest growing instability mode increases exponentially with the index α, with

$$\alpha^2 = \frac{12(1-\nu^2)}{R^2h^2}\frac{D}{\rho h}(\eta^4 - 1), \qquad \eta^2 = \frac{N_0}{N^*} \tag{2.4}$$

Index α_0 of the fastest growing mode in a rod is

$$\alpha_0^2 = \frac{\pi^4 EI}{4\rho F l^4}\eta^4 \tag{2.5}$$

The designations are the same here as in [1]. The instability-mode numbers for (2.4) and (2.5) are respectively

$$m^2 = \frac{2\eta^2[3(1-\nu^2)]^{1/2}}{\pi^2}\frac{L^2}{Rh}, \qquad m_0^2 = \frac{\eta^2}{2} \tag{2.6}$$

The dynamic instability modes and the rates of deflections in a circular cylindrical shell can be determined experimentally by subjecting rods to intensive loading, with the shell simulated by a rod whose parameters are such that $\alpha = \alpha_0$ and $m = m_0$ [see Eqs. (2.4)-(2.6)]. It is to be noted that the equality $\alpha = \alpha_0$ is approximately satisfied when $\eta^2 \gg 1$.

3. Analysis of Instability Modes with the Interaction between Longitudinal and Transverse Motions Taken into Account. In the case of thin-walled shells ($\varepsilon = h/R \ll 1$) the right-hand side of Eq. (1.2) is zero, to the first approximation. For analyzing the solution to Eq. (1.1) with relation (1.3) applicable, we may use the method of nonsteady deformations on a variable interval [5, 6].

The solution to the wave equation (1.2) with the initial conditions (1.4) will be sought in the form

$$N = N_* + \varepsilon N_1 \tag{3.1}$$

The first term N_* here describes the propagation of the boundary mode along a rod (see (1.4)):

$$N_* = N_0 \quad (x < ct), \qquad N_* = 0 \quad (x \geqslant ct) \tag{3.2}$$

The second term is the solution to the nonhomogeneous equation (1.2) with a known right-hand side:

$$N_1 = -\frac{\nu\rho c}{2}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} w_{tt}{}^*(\xi, \tau)\, d\xi \tag{3.3}$$

In order to obtain function w_{tt}^* in (3.3), one extends function w_{tt} oddwise over the entire x axis. If $w_{tt} \equiv 0$ at $x \geq ct$, then $N_1 \equiv 0$ at $x \geq ct$.

Representing the solution to Eq. (1.2) in the form (3.1) is valid until $N_0/N \geq 1$. It must be noted that smallness of the second term in (3.1) is ensured by the fact that the shell is thin-walled.

As is well known, flexural perturbations defined by Eq. (1.1) at $x > ct$ are insignificant. Furthermore, in a more precise formulation of the problem there appear dynamic equations of the Timoshenko kind for a beam, from which it follows that the flexural perturbations propagate at a finite velocity c_* lower than c ($c_* < c$). Therefore, we will analyze the solution to Eq. (1.1) in terms of series defined on a variable interval [6].

After eliminating u_x from Eq. (1.1) with the aid of relation (1.3), we obtain

$$D\left[w_{xxxx} + 12(1-\nu^2)R^{-2}h^{-2}w\right] + (Nw_x)_x + \rho h w_{tt} = f^*(x, t) \tag{3.4}$$

Here $f^*(x, t)$ is a function which depends on the longitudinal force N. If the right-hand side in (1.1) $f(x, t) \equiv 0$ at $x < ct$, then $f^*(x, t) \equiv 0$ at $x > ct$ in (3.4). Consequently, for Eq. (3.4) the problem is formulated on a variable interval. The boundary conditions for this problem are

$$w = w_{xx} = 0 \quad (x = 0), \qquad w = w_x = 0 \quad (x = l = ct) \tag{3.5}$$

The initial conditions are zero-value conditions, namely:

$$w = w_t = 0 \quad (t = 0) \tag{3.6}$$

For expanding the solution to problem (3.4)-(3.6) into a series, we need the instability modes in the shell. We will now examine these modes:

$$\begin{gathered} D\left[W^0_{xxxx} + 12(1-\nu^2)R^{-2}h^{-2}W^0\right] + \lambda W^0_{xx} = 0 \\ W^0 = W^0_{xx} = 0 \quad (x = 0), \qquad W^0 = W_x{}^0 = 0 \quad (x = l) \end{gathered} \tag{3.7}$$

Roots k_j ($j = 1, 2, 3, 4$) of the characteristic equation for (3.7) become purely imaginary at $\lambda \to \infty$:

$$k_{1,2} = \pm ik_0, \; k_{3,4} = \pm ik_1, \; k_0 \sim \sqrt{\lambda}, \; k_1 \to 0 \quad (\lambda \to \infty)$$

Straight calculations show that the natural modes of a rod and of a cylindrical shell converge asymptotically at $\lambda \to \infty$. The asymptote of these modes, $W^\circ = \sin m\pi x/l$, satisfies the conditions of a hinged support at both ends $x = 0, l$: $W^\circ = W_{xx}{}^\circ = 0$.

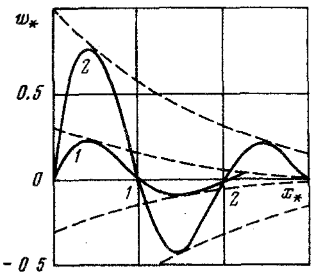

Fig. 1

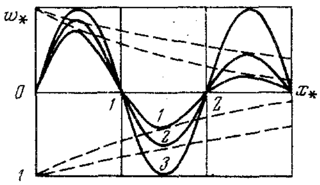

Fig. 2

Let us express the solution to problem (3.4)-(3.6) in the form

$$w = \sum_m W_m(x, t)\, q_m(t) \tag{3.8}$$

where $W_m(x, t)$ are the eigenfunctions of the following problem:

$$D\,[W_{xxxx} + 12(1 - \nu^2) R^{-2}h^{-2}W] + \Lambda\,[1 + \varepsilon s(x, t)]\, W_{xx} = 0$$
$$W = W_{xx} = 0 \quad (x = 0, l), \qquad s(x, t) = N_1 / N_0 \tag{3.9}$$

In order to obtain the eigenfunctions and the eigenvalues of problem (3.9), we make use of the perturbation theory [10]:

$$W_m(x, t) = W_m{}^0(x, t) + \varepsilon W_m^{(1)}(x, t) + \ldots, \quad \Lambda_m = \lambda_m + \varepsilon\lambda_m^{(1)} + \ldots \tag{3.10}$$

Here W_m° and λ_m are the asymptotes of the natural instability modes and of the eigenvalues in problem (3.7). In Eq. (3.9) and in relations (3.10) t is treated as a parameter.

We now insert series (3.8) into Eq. (3.4), and in series (3.10) we disregard derivatives with respect to time which are of the order of ε; this means that all the useful information about the growing instability modes is contained in the function $q_m(t)$. The orthogonality condition for $W_m(x, t)$ on a variable interval yields a system of ordinary differential equations for $q_m(t)$:

$$\rho h q_m'' + \left\{D\left[\left(\frac{m\pi}{l}\right)^4 + \frac{12(1 - \nu^2)}{R^2h^2}\right] - N_0\left(\frac{m\pi}{l}\right)^2 + \varepsilon s^*(t)\right\} q_m = f_m(t) \tag{3.11}$$

Here the small term $\varepsilon s^*(t)$ is related to the variability of the longitudinal force N [see (3.1)] and accounts for the interaction between the various instability modes in the shell (degrees of freedom in the system). For sufficiently thin-walled shells ($\varepsilon \ll 1$) this term may be omitted. From the simplified Eq. (3.11) we find the fastest growing instability modes in a shell [5]. It is easy to show that for these modes the ratio l/m remains constant [see (2.4) and (2.6)]:

$$l^* = \frac{l}{m} = \frac{\pi (Rh)^{1/2}}{\eta\,[12(1 - \nu^2)]^{1/4}} \tag{3.12}$$

The revealed instability mode always occurs in a shell because of the nonlinear term $\varepsilon s^*(t)$ present in Eq. (3.11), even when initial irregularities stimulate the development of one or several instability modes. The effect due to initial irregularities, perturbations in the motion, and variability of force N are all described by the functions $f_m(t)$.

It follows from relations (3.12) that to the first approximation the nodes (the nulls of the flexure function) are fixed in position, and this agrees with the experimental results in [4]. Therefore, we will seek the approximate solution to Eq. (3.4) in the form

$$w(x, \tau) = Q(\tau) W^*(x), \quad W^*(x) = \sin \pi x / l^* \quad (0 \leqslant x \leqslant l), \quad W^*(x) = 0 \quad (x > l) \tag{3.13}$$

Here τ is the real time of action of the compressive force N, i.e.,

$$\tau = t - x / c \tag{3.14}$$

In Eq. (3.4) we now introduce variables x and τ. Disregarding the second-degree terms ($h/l^* \ll 1$), we obtain an equation which is formally identical to Eq. (3.4) with t replaced by τ. We seek a solution to this equation in the form (3.13), applying for this purpose the Bubnov–Galerkin method. In this way, function $Q(\tau)$ is found from the solution to the problem

$$Q'' - \alpha^2 Q = F(\tau), \quad Q = Q' = 0, \quad \tau = 0$$

The solution to this problem is obvious.

Typical buckling modes in shells at various instants of time are shown in Figs. 1 and 2 with the coordinates $w_* = w/Q(t_2)$ and $x_* = x/l^*$; the dashed curves represent the amplitude distribution of a selected instability mode along the shell. In Fig. 1 curve 1 has been plotted for time $t = t_1$, and curve 2 has been plotted for time $t = t_2 = 1.4\ t_1$. The effect of interaction between longitudinal and transverse motions is shown in Fig. 2. At a finite velocity of longitudinal perturbations $c = c_1$ the buckling mode looks like curve 1. For $c = 2c_1$ we have curve 2, and for an infinite velocity $c \to \infty$ we have curve 3 (all curves have been plotted for one and the same instant of time $t = t_2$). The last curve coincides with the buckling mode which has been shown by Lavrent'ev and Ishlinskii in [1] to grow fastest. It is to be noted that the perturbed region ($w \not\equiv 0$), which corresponds to curve 1 in Fig. 2 (and to curve 2 in Fig. 1), propagates till $x = 7l^*$; a transverse motion described by curve 2 in Fig. 2 is observed till $x = 14l^*$; according to [1], the entire shell at once participates in the transverse motion.

Qualitative features of the revealed instability mode agree with experimental results obtained in longitudinal impact loading of rods (Figs. 6 and 7 in [2] and Figs. 1 and 3 in [3]) and cylindrical shells (Figs. 6 and 7 in [4]). A comparison with the experiment is valid only for the time preceding a reflection of the wave. In the experiments in [4] this time was approximately 60 μsec.

LITERATURE CITED

1. M. A. Lavrent'ev and A. Yu. Ishlinskii, "Dynamic instability modes in elastic systems," Dokl. Akad. Nauk SSSR, 64, No. 6 (1949).
2. H. E. Lindberg, "Buckling of a very thin cylindrical shell due to an impulsive pressure," Trans. ASME, Ser. E, J. Appl. Mech., 31, No. 2 (1964).
3. B. M. Malyshev, "Stability of rods under compressive impact," Inzh. Zh. MTT, No. 4 (1966).
4. H. E. Lindberg and R. E. Herbert, "Dynamic buckling of a thin cylindrical shell under axial impact," Trans. ASME, Ser. E, J. Appl. Mech., 88, No. 1, (1966).
5. V. M. Kornev, "Instability modes in an elastic rod under impact," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 3 (1968).
6. L. I. Slepyan, "Analysis of nonsteady deformations with the aid of series defined on a variable interval," Izv. Akad. Nauk SSSR, Mekhanika, No. 4 (1965).
7. V. I. Borisenko and A. I. Klokova, "Transcritical deformation of a cylindrical shell under impact," Prikl. Mekhan., 2, No. 10 (1966).
8. V. M. Kornev, "Instability modes in an elastic circular cylindrical shell under longitudinal impact," in: Transient Deformation Processes in Shells and Plates [in Russian], Izd. AN ÉstSSR, Tallin (1967).
9. A. S. Vol'mir, Stability of Deformable Systems [in Russian], Fizmatgiz, Moscow (1967).
10. E. C. Titchmarsh, Eigenfunction Expansions Associated with Second-Order Differential Equations, Oxford Univ. Press (1958).